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Some properties of schemes in groups theory and Top couples.

Abstract.

Let G be a group, and H a G -group defined by an imbedding map $G \rightarrow H$; in [12] we have defined a topology on a subset of normal subgroups of H , the so-called prime ideals. In this work, we generalize this topology to other classes of groups. We hope that such topologies define new effective tools to tackle well-known problems in groups theory. We study properties of the topology defined in [12], for example we define the notions of functor of points, dimension and outline a Galois theory.

1. Introduction.

Classical algebraic geometry is the study of solutions of polynomial equations defined on commutative rings. The nature of the objects involved here can be very different when one allows the base ring to vary; for example varieties defined over the fields of complex numbers are complex manifolds with singularities, thus, inherit a separated topology from their complex structure, this is not the case of a variety defined over a finite field. The first tools developed in classical algebraic geometry were related to topology or complex geometry on the purpose to study complex schemes. In fact, Serre has shown in the G.A.G.A [8] deep relations between the objects of these theories.

To study the variety defined on more general rings, some tools issued from complex algebraic geometry have been adapted. The most efficient work in that direction has been done by A. Grothendieck with the theory of schemes; he has introduced topological concepts to objects which does not look a priori topologically by generalizing the Zariski topology. This approach has allowed to give new interpretations of many conjectures and often led to their solutions. We can for examples quote the Weil conjectures and more recently the Milnor conjecture.

Many generalizations of the theory of schemes have been achieved for non commutative algebras. In group theory, many classical sets are defined by algebraic-like equations. Thus formally, they are similar to the objects studied in classical algebraic geometry. Recently, Baumslag, Myasnikov and Remeslenikov [2] have initiated the study of varieties of groups. They consider a group G , and $G[X_1, \dots, X_n]$ the free product of G and the free group generated by a set

of cardinality n . For any group H endowed with an injective morphism $G \rightarrow H$, the elements of $G[X_1, \dots, X_n]$ define polynomial functions on H^n which took their values in H . The solutions of these equations define a Zariski-like topology. These authors have adapted tools from the theory of algebraic varieties like Zariski topology and irreducible subsets in groups theory.

In our paper [12], we have generalized the construction of Baumslag, Myasnikov and Remeslennikov by defining a theory of schemes in this context, The purpose of this work is to continue to study this notion; we also generalize it by extending it to others subcategories of the category of groups. We hope that these new concepts will also lead to the solutions of some problems in groups theory by using the topological intuition.

Plan.

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2. Some properties of groups.

Let G be a group, the comma category over G is the category whose objects (H, ϕ_H) are morphisms of groups $\phi_H : G \rightarrow H$. A morphism $f : (H, \phi_H) \rightarrow (H', \phi_{H'})$ of the comma category is a morphism of groups $f : H \rightarrow H'$ such that $f \circ \phi_H = \phi_{H'}$.

We denote by $C(G)$ the subcategory of the comma category such that each object (H, ϕ_H) of $C(G)$ is an object of the comma category such that ϕ_H is injective.

Let (H, ϕ_H) be an object of $C(G)$ and x an element of H . For every element $g \in G$, we denote $\phi_H(g)x\phi_H(g^{-1})$ by x^g . We denote by $G(x)$ the subgroup of H generated by $\{x^g, g \in G\}$. We say that the element $x \in H$ is invertible if and only if $G(x) \cap \phi_H(G) \neq 1$.

Let H be a group, and g, g' elements of H ; the commutator $[g, g']$ of g and g' is the element $gg'g^{-1}g'^{-1}$. If U and V are subgroups of H , $[U, V]$ is the normal subgroup generated by $[u, v], u \in U, v \in V$.

Let H be a group, we denote by $D^0 H$ the group H , suppose defined $D^n H$, for an integer n , then $D^{n+1} H = [D^n H, D^n H]$.

We say that an element x of H distinct of the neutral element is a divisor of zero if and only if there exists an element y of H distinct of the neutral element such that the group of commutators $[G(x), G(y)] = 1$. Remark that the element y in this situation is also divisor of zero.

Proposition 2.1.

The element x of H is a divisor of zero if and only if there exists a non trivial element y of H such that for every element $g \in G$, $[x, y^g] = 1$.

Proof.

Let x be an element of H which is a divisor of zero, and y an element of H such that $[G(x), G(y)] = 1$. The set $U = \{[x, y^g], g \in G\}$ is contained in $[G(x), G(y)]$; we deduce that all elements of U are equal to the neutral element.

Conversely, suppose that all elements of U are equal to the neutral element. Let $u, v \in G$,

$$[x^u, y^v] = uxu^{-1}vyv^{-1}ux^{-1}u^{-1}vy^{-1}v^{-1}.$$

We deduce that:

$$u^{-1}[x^u, y^v]u = xu^{-1}vyv^{-1}ux^{-1}u^{-1}vy^{-1}v^{-1}u = [x, y^{u^{-1}v}]$$

Since $[x, y^{u^{-1}v}] = 1$, we deduce that $[x^u, y^v] = 1$.

Recall that if a, b, c are elements of a group, if a commutes with b and c , then a commutes with b^{-1} and bc , this implies that $G(x)$ commutes with $G(y)$.

Remark.

Let $N_G(x)$ be the normal subgroup generated by $G(x)$, we can find an example of a G -group H such that $[G(x), G(y)] = 1$ and $[N_G(x), N_G(y)] \neq 1$:

Consider $H = SL(4, C)$ the group of linear automorphisms of the 4-dimensional complex vector space C^4 whose determinant is 1. Let e_1, e_2, e_3, e_4 be a basis of C^4 . Let $G = SL(2, C)$ be the subgroup of H such that for every $f \in G$, $f(e_3) = e_3, f(e_4) = e_4$ and $f(e_1)$ and $f(e_2)$ are elements of the vector subspace $Vect(e_1, e_2)$ generated by e_1 and e_2 . Let g be an element of H such that $g(e_1) = e_1, g(e_2) = e_2$ and $g(Vect(e_3, e_4)) \subset Vect(e_3, e_4)$, $[g, G] = 1$, but $[N_G(g), G]$ is not the trivial subgroup: if $f : C^4 \rightarrow C^4$ is defined by $f(e_1) = e_3, f(e_2) = e_4, f(e_3) = e_1$ and $f(e_4) = e_2$, $fgf^{-1} \in G$, and we can choose g such that fgf^{-1} is not in the center of G .

Proposition 2.2.

Suppose that G is a solvable G -group different of $\{1\}$, then any G -group H has a divisor of zero. If H is a finite dimensional simply connected Lie group which does not have zero divisors, then H is simple.

Proof.

Let G be a solvable group different of $\{1\}$; there exists n such that $D^n G$ is not trivial and $D^{n+1} G$ is trivial. For every element $x \in D^n G$ different of its neutral element we have $[G(x), G(x)] = 1$, thus x is a divisor of zero; if H is a G -group, x is also a zero divisor of H .

Suppose that H is a finite dimensional simply connected Lie group which does not have divisors of zero and its radical is not trivial. let R be the radical of H , recall that R is a maximal solvable normal subgroup of H . Let n be an integer such that $D^n R$ is not trivial, and $D^{n+1} R$ is trivial. Since R is a normal subgroup of H , for every element $x \in D^n R$ and every element $g \in G$, $x^g \in D^n R$, this implies that $[G(x), G(x)] = 1$ for every element $x \in D^n R$.

In particular, if x is not the neutral element it is a divisor of zero. This is in contradiction with the fact that we have supposed that G does not have divisors of zero. Thus the radical of H is trivial and H is semi-simple. If H is not simple, then it is the product of two non trivial groups H_1 and H_2 . Let h_1 and h_2 be respectively elements of H_1 and H_2 distinct of the neutral element, we have $[G((h_1, 1)); G((1, h_2))] = 1$. Thus H has divisors of zero.

Remark.

The previous proposition shows that a non commutative nilpotent group G has divisors of zero. In fact, every element in the center of G is a divisor of zero.

Proposition 2.3.

Let k be a field, and G a subgroup of $PSL(n, k)$ suppose that the action of G on the n -dimensional k -projective space Pk^n is reducible, then G has divisors of zero. If G is a connected semi-simple Lie group defined over the complex numbers imbedded in $PSL(n, C)$ or if G is a finite group imbedded in $PSL(n, k)$ where k is a field whose characteristic does not divide the order of G , then G does not preserves the image of a proper linear vector space in $P^n C$ (resp. $P^n k$) if G does not have zero divisors.

Proof.

Suppose that the action of G is reducible. Let G' be the inverse image of the group G by the projection map $p : Sl(n, k) \rightarrow PSL(n, k)$. We can write $k^n = U \oplus V$, where U and V are non trivial vector subspaces of k^n and are stable by G' . Let u' be a non trivial invertible linear map of U and v' be a non trivial invertible linear map of V . Consider the linear morphism u of k^n defined by $u(x) = u'(x)$ if $x \in U$, $u(x) = x$ if $x \in V$. Let v be the linear morphism of k^n defined by $v(x) = x$ if $x \in U$, $v(x) = v'(x)$ if $x \in V$. We have $[G'(u), G'(v)] = 1$. It results that the image of u and v in $PSL(n, k)$ are divisors of zero.

Suppose that G is a connected semi-simple subgroup of $PSL(n, C)$, (resp. G is a finite subgroup of $PSL(n, k)$ such that the order of G and the characteristic of k are prime each other), $p^{-1}(G)$ is semi-simple (resp. finite subgroup) and C^n (resp k^n) under the action of G can be decomposed in the direct summand of irreducible vector spaces. Thus $PSL(n, C)$ (resp. $PSL(n, k)$) has zero divisors if $p^{-1}(G)$ preserves a proper linear subspace.

Definitions 2.1.

A G -domain (H, ϕ_H) of $C(G)$ is an object of $C(G)$ such that H does not have divisors of zero.

A normal subgroup I of H is an ideal of H if and only if $I \cap G = \{1\}$.

A normal subgroup P of H is a prime ideal if and only if:

- P is an ideal distinct of H and

- G/P is a G -domain, this fact is equivalent to saying that for every $x, y \in H$,

$[G(x), G(y)] \in P$ if and only if x is in P or y is in P .

- The G -group H is G -simple if $\{1\}$ is the the only ideal of H .

Proposition 2.4.

Suppose that G is a connected simple Lie group over a field of characteristic zero, and H a connected Lie group in $C(G)$, if H is G -simple then H is simple Lie group.

Proof.

We can write H has a semi-product of a semi-simple group U and a solvable group L by using the Levi decomposition. Without restricting the generality, we can suppose that G is contained in U since G is simple. Since $G \cap L = \{1\}$, L is an ideal. We deduce that L is trivial since the only prime of H is 1. If the subgroup U is not simple, there will exist a simple component of U which intersection with G is trivial since the simple Lie group G is contained in a simple component of H . This fact is impossible since all ideals of H are trivial.

Remark.

The previous result is not true if we don't assume that the groups G and H are connected as shows the following example: Let S_n be the group of permutations of the set of n elements, $n > 4$. The subgroup A_n of permutations of even signature is a simple group. The group S_n is endowed with an A_n -structure. It is well-known that the only normal subgroups of S_n are the trivial subgroup, A_n and S_n . Thus S_n is an A_n -simple group, and S_n is not simple.

Proposition 2.5.

Let k be a field and G a Zariski dense subgroup of $PSL(n, k)$, the G -group $PSL(n, k)$ does not have divisors of zero.

Proof.

Let x be a divisor of zero and y a non trivial element of $PSL(n, k)$ such that for every element $g \in G$, $[x, y^g] = 1$. The equation $[x, y^u] = 1$ is an algebraic equation in $PSL(n, k)$. This implies that the Zariski adherence of G satisfies this equation. Since G is Zariski dense in $PSL(n, k)$, this implies that for every $g \in PSL(n, k)$, $[x, y^g] = 1$. Let x' (resp. y') an element of $SL(n, k)$ above x (resp. above y). Thus $[x', y'^g] = cI_n$ is in the center of $SL(n, k)$ for every $g \in SL(n, k)$. Recall that the elements of the center of $SL(n, k)$ are $c_1 I_1, \dots, c_n I_n$ where c_1, \dots, c_n are the n -roots of unity. The subset G_i of $SL(n, k)$ such that $[x', y'^g] = c_i^{-1} I_n$ is closed, and $SL(n, k)$ is a finite union of the G_i . Since $SL(n, k)$ is irreducible, there exists a i such that $G_i = SL(n, k)$. It results that $[x'^n, y'^g] = I_n$ for every $g \in SL(n, k)$. Let E_h be an eigenvector space of y' associated to the eigenvalue h . Let p be the dimension of E_h , for every vector subspace E of dimension p , there exists an element of $SL(n, k)$ such that $g(E_h) = E$, this implies that E is the eigenvector space of y'^g and x'^n preserves E . We deduce that $x'^n = cI$, $c \in k$. The same argument shows also that $y'^n = dI_n$, $d \in k$. Thus x' and y' are diagonalizable maps. If g is an element of $SL(n, k)$, $g(E_h)$ is an eigenvector space of y'^g associated to h . This implies that $x'(g(E))$ is the eigenvector space of y'^g associated to $c_i^{-1}h$ if $[x', y'^g] = c_i I_n$. Thus x' permutes the eigenvector spaces of y' . Since x' and y' are not homothetic maps since x and y are not trivial, y' has two distinct eigenvector spaces E_1 and E_2 associated respectively to the eigenvalue c_1 and

c_2 ; without restricting the generality, since x' permutes the eigenvector space of y'^g for every $g \in SL(n, k)$, we can suppose that $x'(g(E_1)) = g(E_2)$. This last equality is not possible for every element of $SL(n, k)$.

Proposition 2.6.

Let H be an element of $C(G)$, suppose that G is a normal subgroup of H , then for every ideal ideal I , $[I, G] = 1$.

Proof.

Let g be an element of G , and i an element of I , we have $[u, i] = u i u^{-1} i^{-1}$ is an element of $G \cap I = \{1\}$.

Definitions 2.2.

Let (H, ϕ_H) be an object of $C(G)$, an element x of H is nilpotent of length n if and only if the group $G(n)(x) = [G(x), [G(x), \dots, G(x)]] = 1$ where in the formula $G(x)$ appears n times.

Let H be an element of $C(G)$, the radical $rad_G(H)$ of H is the intersection of all prime ideals of H . We denote by $Com_H(G)$ the subgroup of H such that for every element $h \in Com_H(G)$ and $g \in G$, we have $[g, h] = 1$. The group $Com_H(G)$ is included in every element $P \in Spec_G(H)$ and so is its normalizer $N(Com_H(G))$. This implies that $N(Com_H(G))$ is a subset of $Rad_G(H)$.

Proposition 2.7.

Let P be a prime and x an element of H , suppose that $G(n)(x)$ is contained in P , then x is an element of P . In particular The normal subgroup $Nil(H)$ generated by nilpotent elements of H is contained in P .

Proof.

We show the result recursively. Let P be a prime, suppose that every element u such that $G(n-1)(u) \in P$ is in P . Let x be an element of H such that, $G(n)(x)$ is contained in P . Suppose that x is not in P . Let u be an element of $G(n-1)(x)$, we have $[G(x), u] \subset G(n)(x) \subset P$. Since P is prime, u is an element of P , we deduce that $G(n-1)(x)$ is contained in P , the iteration hypothesis implies that x is an element of P .

Proposition 2.8.

Suppose that G is a simple group, and H is an element of $C(G)$, then every normal subgroup of H is an ideal or contain G .

Proof.

Let I be a normal subgroup of H , $I \cap G$ is a normal subgroup of G ; since G is simple we deduce that $I \cap G = G$ or $I \cap G = \{1\}$.

3. Topologies in group theory.

In this part, we are going to generalize the topology introduced in [12]. Let C be a subcategory of the category of groups, we denote by D a subclass of the class of objects of C which satisfies the following properties:

T1 If G is an object of C , G' is an object of D such that there exists an injective C -morphism $i : G \rightarrow G'$, then $G \in D$.

T2 Let G be an element of D , and I, J normal subgroups of G , $[I, J] = 1$ implies that $I = 1$ or $J = 1$.

Definitions 3.1.

Let H be an element of C , an ideal of H is a normal subgroup I of H such that $H/I \in C$ and the projection $H \rightarrow H/I$ is a morphism of C .

An ideal P of H is a prime if and only if H/P is an object of D and P is distinct of H .

T3 We suppose that the inverse image of an ideal by a morphism of C is an ideal.

A couple (C, D) which satisfies the properties T1, T2 and T3 will be called a Top couple.

Let H be an element of C and I a normal subgroup of H , we denote by $V(I)$ the set of prime ideals of H which contain I .

Proposition 3.1.

Let H be an element of C , and I, J normal subgroups of H , we have:

$$V([I, J]) = V(I) \cup V(J)$$

Let $(I_a)_{a \in A}$ a family of normal subgroups of H , and I_A the normal subgroup generated by $(I_a)_{a \in A}$ we have:

$$V(I_A) = \cap_{a \in A} V(I_a)$$

Proof.

Let P be an element of $V(I) \cup V(J)$, P contains I or J . This implies that P contains $[I, J]$.

Let P be an element of $V([I, J])$ and $p : H \rightarrow H/P$ the natural projection map, we have $[p(I), p(J)] = 1$. Since H/P is an object of D , we deduce that $p(I) = 1$ or $p(J) = 1$. This is equivalent to saying that $I \subset P$ or $J \subset P$.

Let P be an element of $V(I_A)$, for every $a \in A$, we have $I_a \subset I_A \subset P$ this implies $P \in \cap_{a \in A} V(I_a)$.

Let P be an element of $\cap_{a \in A} V(I_a)$, for every $a \in A$, $I_a \subset P$ this implies that $I_A \subset P$ and $P \in V(I_A)$, we deduce that $V(I_A) = \cap_{a \in A} V(I_a)$.

Remarks.

Let (C, D) be a Top couple and H be an object of C , and $\text{Spec}(H)$ the set of prime ideals of H . The set $V(I)$ are the closed subsets of a topology defined on $\text{Spec}(H)$; $V(H)$ is the empty subset, and $V(\{1\}) = \text{Spec}(H)$.

Proposition 3.2.

Let (C, D) be a Top couple and H, H' two objects of C . A C -morphism $f : H \rightarrow H'$ induces a continuous morphism $c(f) : \text{Spec}(H') \rightarrow \text{Spec}(H)$ by setting $c(f)(P) = f^{-1}(P)$.

Proof.

Let P be an ideal of H' , the property T3 implies that $f^{-1}(P)$ is an ideal subgroup of H since the morphism $H/f^{-1}(P) \rightarrow H'/P$ is injective, the property T1 implies that $H/f^{-1}(P) \in D$. This implies that $f^{-1}(P)$ is a prime ideal.

The morphism $c(f)$ is continuous since $c(f)^{-1}(V(I)) = V(f(I))$.

Examples of Top couples.

We are going to present examples of Top couple, we start by the following example which is already presented in Tsemo [12]:

Proposition 3.3.

Let G be a group, and D the class of G -groups which do not have divisors of zero, the couple $(C(G), D)$ is a Top couple.

proof.

We have to show that the couple $(C(G), D)$ satisfies the properties T1, T2 and T3.

If H is a G -group which does not have zero divisors, any G -subgroup of H does not have zero divisors of zero, thus the property T1 is satisfied.

We show now that the couple $(C(G), D)$ satisfies the property T2. Let H be a group which does not have divisors of zero, and I, J two normal subgroups of H such that $[I, J] = 1$. Suppose that $I \neq 1$ and $J \neq 1$. This implies the existence of elements $x \in I$ and $y \in J$ different of the neutral elements. We have $[G(x), G(y)] \subset [I, J] = 1$. This is equivalent to saying that x and y are divisors of zero. This fact is a contradiction to the hypothesis; thus $I = 1$ or $J = 1$.

It remains to show that the couple (C, D) satisfies the property T3. Recall that an ideal I of the element H of $C(G)$ is a normal subgroup I of H such that $I \cap G = 1$, if $f : H \rightarrow H'$ is a G -morphism, and I an ideal of H' , $f^{-1}(I)$ is an ideal of H since the restriction of f on G is injective.

We present now another example of a Top couple. One of the most interesting problem in groups theory is to determine if a group has a finite subgroup which is isomorphic to the free group generated by two elements. Tits Jacques has shown that a subgroup of a linear group is virtually solvable or he contains a free subgroup isomorphic to F_2 . The Top couple that we are presenting now is related to this subject.

Here C is $C(F_2)$ the category whose objects are injective morphisms $F_2 \rightarrow H$. An object $F_2 \rightarrow H$ of C is an object of D if and only if H is a free group. Remark that the rank of an object of D is greater or equal to 2. Let (H, ϕ_H) be an object of $C(F_2)$, an ideal I of H is a normal subgroup I such that $I \cap \phi_H(F_2) = \{1\}$. From this definition, we deduce that the ideal P of H is a prime if and only if H/P is a free group.

Proposition 3.4.

The couple $(C(F_2), D)$ is a Top couple.

Proof.

Let show that $(C(F_2), D)$ satisfies T1. Let $i : H \rightarrow H'$ be an injective F_2 -morphism, the fact that H' is an object of D is equivalent to saying that H' is free, this implies that H is free since a subgroup of a free group is free.

Property T2. Let H be an object of D and I, J ideal of H such that $[I, J] = 1$. Since H is a free group, we deduce that $I = 1$ or $J = 1$.

Property T3 Let $f : H \rightarrow H'$ be an F_2 -morphism, and P an ideal of H' , $f^{-1}(P)$ is an ideal of H since the restriction of f on F_2 is injective, the canonical map $H/f^{-1}(P) \rightarrow H'/P$ is injective and a subgroup of a free subgroup is free.

4. The geometry of G -groups.

In the sequel we are going to consider only the topology mentioned at proposition 3.3.

Proposition 4.1.

Let H be an element of $C(H)$, for every normal subgroups I and J of H , we have $V([I, J]) = V(I \cap J)$.

Proof.

We know that $V([I, J]) = V(I) \cup V(J)$. Since $[I, J] \subset I \cap J$, we have $V(I \cap J) \subset V([I, J])$. Let P be an element of $V([I, J])$, suppose that P is not an element of $V(I \cap J)$, this implies that there exists an element x of $I \cap J$ which is not an element of P . For every element $g \in G$, $[x, x^g] \in [I, J]$ since $x \in I \cap J$. Since P is a G -domain, we deduce that x is an element of P . This is a contradiction, thus P contains $I \cap J$.

Proposition 4.2.

Suppose that G is a solvable group distinct of $\{1\}$, for every object H of $C(G)$, $\text{Spec}_G(H)$ is empty.

Proof.

Let P be a prime ideal of an object H of $V(I)$. Since G is a solvable group distinct of 1, the proof of the proposition 2.2 shows that there exists an element $x \in G$ which is a divisor of zero. This implies that $x \in P$. This is impossible since $P \cap G = \{1\}$.

Remark.

Let G be the trivial subgroup $\{1\}$, any group H is an element of $C(G)$. Let P be a prime ideal of H , for any element $x \in H$, $x \in P$ since $[x, x] = 1$. This implies that $\text{Spec}_G(H)$ is empty since a prime of H is distinct of H .

Let H be an element of $C(G)$, the proposition 3.3 shows that the subsets $V(I)$ where I is a normal subgroup of H are the closed subsets of a topology on the set $\text{Spec}_G(H)$ of prime ideals of H . The empty closed subset is $V(H)$, and the $\text{Spec}_G(H) = V(\{1\})$.

We will say that $\text{Spec}_G(H)$ is an affine scheme.

The structural sheaf.

Let (H, ϕ_H) be an element of $C(G)$, and U an open subset of $\text{Spec}_G(H)$, we denote by $\text{Rad}(U)$ the intersection of the elements of U . We define on $\text{Spec}_G(H)$ the structural presheaf:

$$U \longrightarrow P_H(U) = H/\text{Rad}(U)$$

Let U and V be two open subsets of $\text{Spec}_G(H)$ such that $U \subset V$; $\text{Rad}(V) \subset \text{Rad}(U)$, this implies the existence of a map $r_{U,V} : P_H(V) = H/\text{Rad}(V) \rightarrow P_H(U) = H/\text{Rad}(U)$ which is the restriction map of the presheaf P_H .

The structural sheaf of O_H of $\text{Spec}_G(H)$ is the sheaf associated to the presheaf P_H .

Recall that if x is an element of $\text{Spec}_G(H)$ and $U(x)$ the set of open subsets of $\text{Spec}_G(H)$ which contains x , the stalk $P_{H,x}$ of x is the inductive limit $P_{H,x} = \lim_{V \in U(x)} P_H(V)$. The etale space E_{P_H} of P_H is the topological space which is the disjoint union $\{P_{H,x}, x \in \text{Spec}_G(H)\}$ endowed with the topology generated by $\{(s_x)_{x \in U}, s \in P_H(U)\}$ where U is any open subset of $\text{Spec}_G(H)$. The sheaf O_H is a sheaf of continuous sections of the canonical morphism $p : E_{P_H} \rightarrow \text{Spec}_G(H)$ which sends an element of $P_{H,x}$ to x .

Definition 4.1.

Let (X, O_X) and (Y, O_Y) be two topological spaces X and Y respectively endowed with the sheaves O_X and O_Y . A morphism $(f, f') : (X, O_X) \rightarrow (Y, O_Y)$ is defined by a continuous map $f : X \rightarrow Y$ and a morphism of sheaves $f' : O_Y \rightarrow f^{-1}O_X$ which commutes with the restriction maps.

Proposition 4.3.

Let (H, ϕ_H) and (L, ϕ_L) elements of $C(G)$ with trivial radicals, there exists a natural bijection between morphisms: $(f, f') : (\text{Spec}_G(H), O_H) \rightarrow (\text{Spec}_G(L), O_L)$ and $\text{Hom}_G(L, H)$.

Proof.

Let $(f, f') : (\text{Spec}_G(H), O_H) \rightarrow (\text{Spec}_G(L), O_L)$ be a morphism. We associate to (f, f') the G -morphism $f'(\text{Spec}_G(L)) : O_L(\text{Spec}_G(L)) = L \rightarrow O_H(\text{Spec}_G(H)) = H$.

Let $u : L \rightarrow H$ be a G -morphism of $C(G)$, then u induces a continuous map $(f_u, f'_u) : (\text{Spec}_G(H), \phi_H) \rightarrow (\text{Spec}_G(L), \phi_L)$ defined as follows:

Let P be an element of $\text{Spec}_G(H)$, then $f_u(P) = u^{-1}(P)$. The proposition 3.3 shows that f_u is well defined. Let V be an open subset of $\text{Spec}_G(L)$, the composition by u induces a morphism $f'_u(V) : O_L(V) \rightarrow f^{-1}(O_H)(V)$.

The correspondences defined above between $\text{Hom}_G(L, H)$ and the set of morphisms between $(\text{Spec}(H), O_H)$ and $(\text{Spec}(L), O_L)$ are inverse each others.

Definition 4.2.

A topological manifold X is an G -scheme if and only if there exists a covering $(U_i)_{i \in I}$ of X and a sheaf O_X on X , such that for every $i \in I$, there exists an affine G -scheme $(\text{Spec}_G(H_i), O_{H_i})$ and an homeomorphism of sheafed spaces $f_i : (\text{Spec}_G(H_i), O_{H_i}) \rightarrow (U_i, O_{U_i})$, where O_{U_i} is the restriction of O_X to U_i .

Proposition 4.4.

Let (H, ϕ_H) be an element of $C(H)$, and $Nil(H)$ the normal subgroup of H generated by its nilpotent elements. The projection morphism $p' : H \rightarrow H/Nil(H)$ induces an homeomorphism between $p : Spec_G(H/Nil(H)) \rightarrow Spec_G(H)$.

Proof.

Let P be a prime ideal of $H/Nil(H)$, $p'(p'^{-1})(P) = P$. This implies that p is injective. Let Q be a prime ideal of H , $p'^{-1}(p'(Q)) = Q$ since Q contains $Nil(H)$. This implies that p is surjective thus bijective. Let I be an ideal of $H/Nil(H)$, $p(V(I)) = V(p'^{-1}(I))$, this implies that p is open. We conclude that p is an homeomorphism.

Definition 4.3.

Let G and L be two elements of $C(G)$ with trivial radical. A morphism $(f, f') : (Spec_G(H), O_H) \rightarrow (Spec_G(L), O_L)$ is of finite type if and only if the L -group H is the quotient of a free group $L[x_1, \dots, x_n]$ by an ideal generated by a finite subset for the L -structure of H induced by (f, f') .

The smallest n such that H is the quotient of $L[x_1, \dots, x_n]$ by an ideal of $L[x_1, \dots, x_n]$ generated by a finite subset is called the L -rank of G .

Suppose that G is a domain and the radical of H is trivial, we say that $(Spec_G(H), O_H)$ is of finite type if and only if the canonical morphism $(Spec_G(H), O_H) \rightarrow (Spec_G(G), O_G)$ is a morphism of finite type.

We endow now the category of affine schemes with a product:

Proposition 4.5.

Let H and K be two elements of $C(G)$, the affine scheme $Spec_G(H *_G K)$ is the categorical product of $Spec_G(H)$ and $Spec_G(K)$.

Proof.

A couple of morphisms $f : Spec_G(L) \rightarrow Spec_G(H)$ and $g : Spec_G(L) \rightarrow Spec_G(K)$ is defined by a couple of morphisms of G -groups $f' : H \rightarrow L$ and $g' : K \rightarrow L$ which induces a unique morphism $(f', g') : H *_G K \rightarrow L$ such that $f' = (f', g') \circ i_H$ and $g' = (f', g') \circ i_K$ where $i_H : H \rightarrow H *_G K$ is the map defined by $i_H(h) = h *_G 1$. This is equivalent to saying that $Spec_G(H *_G K)$ is the product of $Spec_G(H)$ and $Spec_G(K)$.

Proposition 4.6.

The category of affine G -schemes has sums.

Proof.

Let H and K be two elements of $C(G)$, we can endow the product of these groups $H \times K$ with the diagonal action of G ; Let $f : Spec_G(H) \rightarrow Spec_G(L)$ and $g : Spec_G(K) \rightarrow Spec_G(L)$ be morphisms of G -schemes. These morphisms defined respectively by G -morphisms: $f' : L \rightarrow H$ and $g' : L \rightarrow K$ which induce a G -morphism $(f', g') : L \rightarrow H \times K$, which defines the sum $(f, g) : Spec_G(H \times K) \rightarrow Spec_G(L)$.

Definitions 4.4.

Let H be an element of $C(G)$, we define a functor $c_H : C(G) \rightarrow C(H)$ which associates to L the amalgamated sum $H *_G L$. We say that the H -scheme

$\text{Spec}_H(H *_G L)$ is obtained from L by a change of the basis. In particular if H is an algebraic closure G_{al} of the group G , $\text{Spec}_{G_{al}}(G_{al} *_G L)$ will be called the geometric scheme associated to H .

Definitions 4.5.

Recall that an irreducible topological set is a set which cannot be the union of two proper distinct closed subsets.

Let (H, ϕ_H) be an element of $C(G)$. The normal subgroup I of H is a radical, if I is the intersection of all the primes ideals which contains I .

Proposition 4.7,

Let (H, ϕ_H) be an element of $C(H)$, if I a prime ideal of H , then $V(I)$ is irreducible. Conversely, suppose that I is a radical ideal, then if $V(I)$ is irreducible I is a prime ideal.

Proof.

Suppose that I is a prime ideal and $V(I)$ is the union of the proper closed subsets $V(J)$ and $V(K)$. Since I is a prime ideal, I is an element of $V(I)$; thus I is an element of $V(J)$ or I is an element of $V(K)$. If I belongs to $V(J)$; this implies that I contains J , thus $V(I) \subset V(J)$. This is a contradiction with the fact that $V(J)$ is a proper closed subset of $V(I)$. The same argument shows that I cannot belong to $V(K)$.

Suppose now that $V(I)$ is an irreducible closed subset and I is a radical ideal. Let x and y be elements of H such that $[G(x), G(y)] \subset I$. For every element $P \in V(I)$, $[G(x), G(y)] \subset P$, this implies that $x \in P$ or $y \in P$. Let V_x the subset of elements of $V(I)$ which contains x and V_y the subset of elements of $V(I)$ which contains y , $V(I)$ is the union of $V(x)$ and $V(y)$. Let $N(x)$ be the minimal subgroup of H which contains x , if P is an ideal which contains x , $N(x) \subset P$. This implies that $V(x) = V(I) \cap V(N(x))$ and $V(y) = V(I) \cap V(N(y))$. Since $V(I)$ is irreducible, we deduce that $V(I) \cap V(N(x)) = V(I)$ or $V(I) \cap V(N(y)) = V(I)$. Suppose that $V(I) \cap V(N(x)) = V(I)$, this implies that $V(I) \subset V(N(x))$ and for every prime ideal $P \in V(I)$, $x \in P$, since I is a radical ideal, $x \in I$. By the same argument, we deduce that if $V(I) \cap V(N(y)) = V(I)$, $y \in I$.

Corollary 4.1.

Suppose that G does not have divisors of zero, $\text{Spec}_G(H)$ is irreducible if and only if the radical of H is a prime ideal.

Proof.

Recall that the radical $\text{Rad}_G(H)$ is the intersection of all the prime ideals of $\text{Spec}_G(H)$. Thus by definition, $\text{Rad}_G(H)$ is a radical ideal, we can apply proposition 4.7.

Remark.

If $\text{Rad}_G(H)$ is a prime, it is a generic point of $\text{Spec}_G(H)$, that is, its adherence is $\text{Spec}_G(H)$.

5. Functor of points.

Let H be an element of $C(G)$. Consider the n -uple $[h] = (h_1, \dots, h_n)$ of H^n ; it induces a G -morphism:

$$u_{[h]} : G[X_1, \dots, X_n] \rightarrow H$$

defined by $u_{[h]}(X_i) = h_i$, for every element $f \in G[X_1, \dots, X_n]$, we set $f_H([h]) = u_{[h]}(f)$, we have thus defined a morphism $f_H : H^n \rightarrow H$.

Let S be a subset of $G[X_1, \dots, X_n]$ in their paper [2], Baumslag and al defined the subset:

$$V_H(S) = \{[h] = (h_1, \dots, h_n) \in H^n : \forall f \in S, f_H([h]) = 1.\}$$

We are going to present this notion with the concept of functor of points.

Definition 5.1.

We denote by $Spec_G^0$ opposite to the category $Spec_G$ of affine G -schemes. For every object H of $C(G)$, the functor of points $h_{Spec_G(H)} : Spec_G^0 \rightarrow Sets$ is the functor defined by: $h_{Spec_G(H)}(Spec_G(K)) = mor_G(Spec_G(K), Spec_G(H)) = Mor_G(H, K)$.

In particular, if H is a scheme of finite type, H is the quotient of $G[X_1, \dots, X_n]$ by an ideal I , an element $f \in Mor_G(H, K)$ is defined by a n -uple $(k_1, \dots, k_n) \in K^n$ such that for every $f \in I$, $f(k_1, \dots, k_n) = 1$. Thus the elements of $Mor_G(H, K)$ in this situation correspond to the elements of the closed subset $V_G(K)$ defined in [2], where S is replaced by I .

6.1 Noetherian G -schemes.

The elements of $G[X_1, \dots, X_n]$ are often called equations in n -variables over G . Every set of equations E generates a normal subgroup I_E of $G[X_1, \dots, X_n]$.

Definition 6.1.

We say that the group G is Noetherian if and only if for every set of equations E , there exists a finite subset E_0 of E such that $V_G(I_E) = V_G(I_{E_0})$.

The problem of the existence of a solution of a finite set of equations defined on a group G has been studied by many authors. This situation is analog the study of the roots of polynomial equations in field theory. In fact, Scott [7] has defined a notion of algebraically closed group that we recall.

Definition 6.2.

A set E of equations and inequations defined on a group G is consistent, if there exists an embedding $G \rightarrow H$ such that E has a solution in H .

An example of a set of equations and inequations which is not consistent is: $x^2 = 1, x^3 = 1, x \neq 1$.

Definitions 6.3.

A group is algebraically closed if and only if every finite set of equations consistent defined on G has a solution in G .

Scott has shown that every group can be imbedded in an algebraically closed group. We show now the following proposition (See also Baumslag and al [2] p. 62)

Proposition 6.1.

Let G be an algebraically closed group, then for every finitely generated maximal M ideal of $G[X_1, \dots, X_n]$, there exists elements a_1, \dots, a_n in G such that M is generated by $\{X_1 a_1^{-1}, \dots, X_n a_n^{-1}\}$.

Proof.

Let H be the quotient $H = G[X_1, \dots, X_n]/M$, and x_1, \dots, x_n , the image of X_1, \dots, X_n by the natural projection $G[X_1, \dots, X_n] \rightarrow H$. Then (x_1, \dots, x_n) is a solution of every equation defined by elements of M . Since G is algebraically closed and M is finitely generated, there exists an element (a_1, \dots, a_n) in G^n which is a solution of the equations defined by elements of M . This implies that the ideal generated by $X_1 a_1^{-1}, \dots, X_n a_n^{-1}$ contains M henceforth this ideal is equal to M since M is maximal.

Definition 6.4.

A topological set is Noetherian if and only if every descending chain of closed subspaces $Y_1 \supseteq Y_2 \dots \supseteq Y_i \dots$ stabilizes. This means that there exists n such that $Y_i = Y_n$ for every $n \geq i$.

Let H be an object of $C(G)$, H is a Noetherian group if and only if every family of ideals $I_1 \subset I_2 \subset \dots \subset I_n \dots$ stabilizes, this is equivalent to saying that there exists an i such that for every $n \geq i$, $I_n = I_i$.

Remark that $\text{Spec}_G(H)$ is Noetherian if and only if H is a Noetherian group.

Proposition 6.2.

The object H of $C(G)$ is Noetherian if and only if every ideal of H is finitely generated.

Proof.

Suppose that H is Noetherian, let I be an ideal of H , consider the set A whose objects are finitely generated ideals contained in I ordered by the inclusion. Every totally ordered family $(I_l)_{l \in L}$ of A has a maximal element. If not we can construct a subfamily $(I_n)_{n \in \mathbb{N}}$ of $(I_l)_{l \in L}$ such that I_n is strictly contained in I_{n+1} . This is in contradiction with the fact that H is Noetherian. Let M be a maximal element of A , $M = I$, if not there exists an element x of I which is not in M and the ideal generated by M and x contains strictly M .

Suppose that every ideal of H is finitely generated. Let $I_0 \subset I_2 \dots \subset I_n \dots$ be an ascending sequence of ideal, $\bigcup_{n \in \mathbb{N}} I_n = I$ is a finitely generated ideal by x_1, \dots, x_p . The element x_i belongs to $I_{m(i)}$. Let m be the maximum of $m(i)$, $I = I_m$.

Proposition 6.3.

Suppose that $\text{Spec}_G(H)$ is a Noetherian topological space, then the radical of H is the intersection of a finite number of primes ideals. In particular if the intersection of all the elements of $\text{Spec}_G(H)$ is the neutral element, then there

exists a finite number of elements of $\text{Spec}_G(H)$ whose intersection is the neutral element.

Proof.

We know that a Noetherian topological space is the union of a finite number of irreducible closed subsets. This implies that there exists a finite number of primes ideals P_1, \dots, P_n such that $\text{Spec}_G(H) = \bigcup_{i=1, \dots, n} V(P_i)$. The proposition 4.1 implies that $\text{Spec}_G(H) = V(\bigcap_{i=1, \dots, n} P_i)$. This implies that every prime ideal P of H contains $\bigcap_{i=1, \dots, n} P_i$, thus $\bigcap_{i=1, \dots, n} P_i \subset \text{Rad}_G(H)$. It results that $\text{Rad}_G(H) = \bigcap_{i=1, \dots, n} P_i$ since $\text{Rad}_G(H)$ is the intersection of all the elements of $\text{Spec}_G(H)$.

7. Dimension of affine schemes.

We want to introduce the notion of dimension of an affine schemes. In commutative algebra, the dimension of a ring can be defined to be its height that is, the maximal length of chains $0 \subset P_1 \subset \dots \subset P_n$ of prime ideals such that P_i is distinct of P_{i+1} . We can adopt the same definition here for the prime ideals of elements of H for every object H of $C(G)$. But the following proposition shows that the height of $G * Z$ can be infinite.

Proposition 7.1.

*Let G be a group which does not have divisors of zero, then the height of $G * Z$ is infinite.*

Proof.

Let I_n be the ideal of $G * Z$ generated by 2^n , $(G * Z)/I_n = G * (Z/2^n)$. Suppose that $G * Z/2^n$ have divisors of zero x, y . For every $g \in G$, we have $[x, y^g] = 1$, x and y cannot be in a subgroup conjugated to $G * 1$ since G does not have zero divisors; x, y cannot be in a subgroup conjugated to $1 * Z$ in this case, $[x, y^g]$ is not trivial for every non trivial element of $g \in G$. If x is not in a conjugated of $G * 1$ and $1 * Z$, then [5] page 187 shows that for every $g \in G$, y^g and x are the power of the same element; this is impossible. We conclude that I_n is a prime, thus the sequence $I_n \subset I_{n-1} \subset \dots \subset I_0$ is an infinite sequence of distinct prime ideals.

In classical algebraic geometry, the dimension of a scheme is also the the transcendence dimension of its field of fraction. The corresponding notion here is the G -rank:

Definition 7.1.

Let H be an object of $C(G)$, a family of G -generators of H is a family of elements $(x_i)_{i \in I}$ of H such that G and $(x_i)_{i \in I}$ generate H . The G -rank of H is the minimal cardinal of the set I such that there exists a family of generators $(x_i)_{i \in I}$ of H .

We have the following result:

Proposition 7.2.

Let G be a simple group, and G^n the product of n copies of G endowed with the G -structure defined by the embedding of G as the first factor, $(G, 1, \dots, 1)$ then $\text{Spec}_G(H)$ contains only one element the prime $(1, G, \dots, G)$ and the $\text{rank}_G(G^n) = n - 1$.

Proof.

Let I be an ideal of G^n , suppose that I contains an element $u = (g_1, \dots, g_n)$ such that g_1 is different of the neutral element of G . For every $g \in G$, let $i_g = (g, 1, \dots, 1)$, we have: $i_g u i_g^{-1} = (g g_1 g^{-1}, g_2, \dots, g_n) \in I$. Since G is simple, there exists $g \in G$ such that $g g_1 g^{-1} \neq g_1$. This implies that $u^{-1} i_g u i_g^{-1}$ is a non trivial element of G . This is a contradiction since I is an ideal, thus I is contained in $\{1\} \times G^{n-1}$. If I is distinct of $\{1\} \times G^{n-1}$, there exists an element $v \in \{1\} \times G^{n-1}$ who does not belong to I . Let $p : G^n \rightarrow G^n/I$ be the natural projection, for every element $g \in G$, we have $[g, v] = 1$ thus $[p(g), p(v)] = 1$. This implies that G^n/I is not a domain, thus I is not a prime.

Remark.

Suppose that the center of G is not trivial, then for every element of $C(G)$, $\text{Spec}_G(H)$ is empty and the G -rank of H needs not to be 0 or $-\infty$.

Definition 7.2.

Let H be an object of $C(G)$:

- if $\text{Spec}_G(H)$ is empty, the dimension of $\text{Spec}_G(H)$ is $-\infty$.

if $\text{Spec}_G(H)$ is not empty, the dimension of $\text{Spec}_G(H)$ is the G -rank of $H/\text{Rad}_G(H)$ the quotient of H by the intersection $\text{Rad}_G(H)$ of all the elements of $\text{Spec}_G(H)$.

Remarks.

If H is a G -domain, then $\text{Rad}_G(H) = \{1\}$ and the dimension of $\text{Spec}_G(H)$ is $\text{rank}_G(H)$.

Let G be a group without zero divisors, and $H = G[x_1, \dots, x_n]$, H is a G -domain and the dimension of H is n .

If H is a G -domain, then dimension of $\text{Spec}_G(H) = 0$ is equivalent to saying that H is isomorphic to G .

Let $H = G^n$ endowed with the G -structure defined by the imbedding $(G, 1, \dots, 1)$ of G in H , then $\text{Rad}_G(H) = (1, G, \dots, G)$ and $H/\text{Rad}_G(H) \simeq G$; this implies that the dimension of $\text{Spec}_G(H) = 0$.

We know that a if H is G -simple, then $\text{Spec}_G(H)$ contains only one element, but the G -rank of H is not necessarily 1. But if G is algebraically closed, we have the following result:

Proposition.

Suppose that G is algebraically closed and H a G -domain of finite type which is G -simple, then $H = G$, thus $\dim_G(H) = 0$.

Proof.

There exists a positive integer n and an ideal I of $G[x_1, \dots, x_n]$ such that $H = G[x_1, \dots, x_n]/I$. Since G is algebraically closed, there exists a maximal ideal

M (defining by a solution of the equations defined by a finite set of generators of I containing I) such that $G[x_1, \dots, x_n]/M = G$. Let M' be the image of M by the projection map $G[x_1, \dots, x_n] \rightarrow H$, M' is an ideal of H , since H is G -simple, we conclude that $M' = \{1\}$ thus $M = I$. We deduce that $H = G$ since $H = G[x_1, \dots, x_n]/M$.

Let H be a group, the co-rank of n is the maximal integer n such that there exists a surjective morphism: $H \rightarrow F[x_1, \dots, x_n]$. We say that the G -co-rank of the object n in $C(G)$ is n if n is the maximal integer such that there exists a G -morphism: $H \rightarrow G[x_1, \dots, x_n]$.

Contrary to the rank, a class of ideals allows to express geometrically the idea of the co-rank,

Definition.

Let H be an element of $C(G)$, a G -free ideal I of H is an ideal such that $H/I = G[x_1, \dots, x_n]$.

The co-dimension of $\text{Spec}_G(H)$ is the maximum integer n such that there exists an ideal I such that $H/I = G[x_1, \dots, x_n]$.

Proposition 7.4.

Let H be an element of $C(G)$, the co-dimension of $\text{Spec}_G(H)$ is the length of the maximal sequence $P_0 \subset P_2 \subset \dots \subset P_n$ such that for $i = 1, \dots, n$ P_i is G -free and P_{i+1} contains strictly P_i .

Proof.

Suppose that the co-dimension of $\text{Spec}_G(H)$ is n , there exists a surjective G -morphism $f : H \rightarrow G[x_1, \dots, x_n]$. We denote by P_0 the ideal $f^{-1}(1)$ of H . Let I_l be the normal subgroup of $G[x_1, \dots, x_n]$ generated by x_1, \dots, x_l , and for $i = 1, \dots, n$, $P_i = f^{-1}(I_i)$. The family of ideal $(P_i)_{i \in \{0, \dots, n\}}$ is a chain of free ideals of length n such that P_i is strictly included in P_{i+1} .

Let $P_0 \subset P_2 \subset \dots \subset P_n$ be a maximal chain of G -free ideals such that P_{i+1} contains strictly P_i . Remark that the G -co-rank of H is inferior or equal to n , since if the co-rank is superior to n the first stage of the proof shows that we can construct a chain of G -free prime ideals of length strictly greater than n and whose G -co-rank is strictly decreasing. Thus it is enough to show that $G/P_0 \simeq G[x_1, \dots, x_n]$.

Suppose that the G -rank of G/P_0 is strictly inferior to n ; this implies that there exists, $i \leq n$ such that $\text{co-rank}(P_i) = \text{co-rank}(P_{i+1})$. Let $l_i : H \rightarrow H/P_i$ be the natural projection, H/P_{i+1} is the quotient of H/P_i by $l_i(P_{i+1})$. Write $H/P_i = G[x_1, \dots, x_{n_i}]$ and $H/P_{i+1} = G[y_1, \dots, y_{n_i}]$. The quotient of $G[x_1, \dots, x_{n_i}]$ by the normal subgroup generated by G is isomorphic to the free group $F[x_1, \dots, x_{n_i}]$. The quotient of $G[y_1, \dots, y_{n_i}]$ by the normal subgroup generated by G is isomorphic to the quotient of $F[x_1, \dots, x_{n_i}]$ by the normal subgroup generated by the image of $l_i(P_{i+1})$ in $F[x_1, \dots, x_{n_i}]$. Since this group is isomorphic to the free group of rank n_i $F[y_1, \dots, y_{n_i}]$, we deduce that $l_i(P_{i+1})$ is included in the normal subgroup of $G[x_1, \dots, x_{n_i}]$ generated by G . This in contradiction with the fact that the quotient of $G[x_1, \dots, x_{n_{i+1}}]$ by the image of $l_i(P_{i+1})$ is a free product of G and a free group.

The G -free ideals of H generate a topology as shows the following proposition:

Proposition 7.5.

Let G be a group which does not have divisors of zero, and Let H be an element of $C(G)$, for every normal subgroup I of H , denote $V_F(I) = \{P : P \text{ is a } G\text{-free prime ideal and } I \subset P\}$, then for every normal subgroups I, J of H , $V_F(I) \cup V_F(J) = V_F([I, J])$. Let $(I_a)_{a \in A}$ be a family of normal subgroups of H . Denote by I_A the normal subgroup generated by $(I_a)_{a \in A}$, then $V_F(I_A) = \bigcap_{a \in A} V_F(I_a)$.

Proof.

Firstly we show that $V_F(I) \cup V_F(J) = V_F([I, J])$. Let P be an element of $V_F(I) \cup V_F(J)$, P contains I or J this implies that P contains $[I, J]$. Let P be an element of $V_F([I, J])$. Suppose that P is neither an element of $V_F(I)$ nor an element of $V_F(J)$. This implies that there exists an element $x \in I$ and $y \in J$ which are not in P . For every $g \in G$, $[x, y^g] \in P$. Let $p : H \rightarrow H/P$ be the natural projection, we have $[p(x), p(y)^g] = 1$. Since H/P is G -free, we deduce that x and y are elements of G , this is in contradiction with the fact that G does not have divisors of zeros.

Now we show that $V(I_A) = \bigcap_{a \in A} V(I_a)$. Let P be an element of $V(I_A)$, since $I_a \subset I_A \forall a \in A$, $P \subset V(I_a) \forall a \in A$. Let P be an element of $\bigcap_{a \in A} V(I_a)$, $\forall a \in A$, $I_a \subset P$, this implies that $I_A \subset P$ and $P \in V(I_A)$.

Remark.

Let G be a group which does not have zero divisors, and H an object of $C(G)$, the previous proposition shows that the family of subspaces $V_F(I)$ where I is any normal subgroup in H defines a topology on the space $\text{Spec}_G^F(H)$ of G -free prime ideals of H . But if $f : H \rightarrow H'$ is a morphism between the elements H and H' of G and P is an element of $\text{Spec}_G^F(H)$, $f^{-1}(P)$ is not necessarily a G -free prime of H . For example let $G = PSL(n, Z)$ and $H' = PSL(n, Z)[X]$. Let H be the subgroup of H' generated by $PSL(n, Z)$ and XNX^{-1} where N is a nilpotent subgroup of $PSL(n, Z)$. The canonical imbedding $i : H \rightarrow H'$ is a G -morphism. The normal trivial subgroup $I = \{1\}$ of H' is a prime ideal of H , but the quotient of H by $i^{-1}(I)$ is not G -free, since H is not G -free.

The theorem of Kurosh says that a subgroup of a free product of groups $H_1 * H_2$ is the free product of a free group and groups isomorphic to subgroups of H_1 and H_2 . Thus if we choose $G = F_n$ $n \geq 2$, a G -morphism $f : H \rightarrow H'$ induces a morphism $\text{Spec}_G^F(H') \rightarrow \text{Spec}_G^F(H)$ of G -free spectra. In fact in this case the topology is defined by a Top couple (see definition 3.1).

Definition 7.4.

Let G be a group without divisors of zero, and H an element of $C(G)$; an element P of $\text{Spec}_G(H)$ is a G -point if and only if $H/P = G$.

Remark.

If P is a G -point, the restriction of the projection $p : H \rightarrow H/P$ to G is an isomorphism, thus this extension splits.

Proposition 7.6.

Let H be a G -domain if G is a normal subgroup of H , for every prime P of H , we have $[G, P] = 1$. If there exists a G -point P in H , then $H = P \times G$.

Proof.

Let P be a prime ideal. We have $[P, G] \subset P \cap G = \{1\}$ since P is an ideal and G is a normal subgroup of H .

If P is a G -point of H , then H is an extension of G by P which splits, since $[P, G] = 1$ we deduce the result.

8. Galois theory.

Let G be a group, a polynomial of G is an element of $G[X]$. The Galois theory in classical field theory enables to study the solutions of polynomial functions defined on a field. One of the property essential in that study is the existence of factorization: let P be a polynomial defined on a field, and a a root of P , we can write $P = (X - a)Q$; if $\deg(P)$ is the degree of P , we have: $\deg(Q) = \deg(P) - 1$. Here we do not have factorization system, in fact we can have a polynomial which has an infinite numbers of roots: Let $G = Gl(2, C)$ be the group of automorphisms of the complex plan; the solutions of the equation defined on G by $X^2 = 1$ is the set of symmetries of the complex plan. We can introduce anyway some notions related to Galois theory.

Definitions 8.1.

Let G be a group, and H an element of $C(G)$, an element $a \in H$ is algebraic over G if a is the root of a polynomial whose coefficients are in G .

An element H of $C(G)$ is an algebraic extension of G if and only if every element of H is algebraic over G .

We say that an algebraic extension of G is finite if and only if the action of G on H by left multiplications has a finite number of orbits

Proposition 8.1.

Suppose that $G = \{1\}$, then a group H is an algebraic extension of G if and only if every element of H has a finite order. In particular a group is a finite extension of $\{1\}$ if and only if it is a finite group.

Proof.

If $G = \{1\}$, then $G[X] = F[X] = Z$ and the polynomials of $\{1\}$ are X^n . A group H is algebraic over $\{1\}$ if and only if for every h in H , there exists an element P of $F[X]$ such that $P(h) = 1$. This is equivalent to the existence of an integer n such that $h^n = 1$.

Suppose that H is a finite group. Let n_H be the order of H , for every element $h \in H$, we have $h^{n_H} = 1$; this implies that H is algebraic over $\{1\}$. The left action of $\{1\}$ has finite orbit since H is finite. Conversely, if the left action of $\{1\}$ on H has finite orbits, then H is a finite group.

Remark.

An algebraic extension H of $\{1\}$ is not necessarily finite even if H is finitely generated. In 1904 Burnside asks the question whether every finitely generated

group for which every element has a finite order is finite. There are groups which have been constructed answering negatively the problem of Burnside.

Proposition 8.2.

Let H be an element of $C(G)$, suppose that H is the union of a finite number of orbits for the action of G on H defined by the left multiplication, then H is algebraic over G . In particular if G is a normal subgroup of H such that G/H is finite, H is algebraic over G .

Proof.

Let x be an element of H ; if H is the union of a finite number of orbits for the left action of G , there exists distinct integers n and m with $n < m$ such that x^n and x^m are in the same orbit. This is equivalent to saying that there exists $h \in H$, $g_1, g_2 \in G$ such that $x^n = g_1 h$ and $x^m = g_2 h$, we deduce that $h = g_1^{-1} x^n = g_2^{-1} x^m$. It results that $g_1 g_2^{-1} x^{m-n} = 1$. Thus x is algebraic over G .

Proposition 8.3.

Let H be an algebraic extension of G , then the G -co-rank of H is strictly inferior to 1.

Proof.

Suppose that the G -co-rank of H is superior or equal to 1. There exists a G -surjection: $s : H \rightarrow G[x_1, \dots, x_n]$. Let y_1, \dots, y_n a family of algebraic elements of H which generate H . There exists $i \in \{1, \dots, n\}$ such that $s(y_i)$ is not in G . Since y_i is algebraic, there exists a G -polynomial P such that $P(y_i) = 1$. We also have $P(s(y_i)) = 1$. This is impossible since $s(y_i)$ is a non trivial words written with elements of G and with elements of the free group generated by x_1, \dots, x_n .

Proposition 8.4.

Let H be an element of $C(G)$ and K an element of $C(H)$, suppose that H is a finite algebraic extension of G and K is a finite algebraic extension of H ; then K is a finite algebraic extension of G .

Proof.

To show this proposition it is enough to show that the action of G on K has a finite number of orbits by applying proposition 8.2.

Since H is a finite algebraic extension of G and K a finite algebraic extension of H , we know that the orbit spaces H/G and K/H are finite; we denote by h_1, \dots, h_m elements of H such that the union of gh_1, \dots, gh_m is H , and by k_1, \dots, k_n elements of K such that the union of hk_1, \dots, hk_n is K . Let x be an element of K , there exists $1 \leq i \leq n$ such that $x \in Hk_i$, since H is the union of gh_1, \dots, gh_m , we deduce that $x \in \{gh_1 k_i, \dots, gh_m k_i\}$. This is equivalent to saying that the union of $gh_j k_i$ $1 \leq j \leq m$, $1 \leq i \leq n$ is K .

Definition 8.2.

Let H be an algebraic extension of G , the Galois group $Gal_G(H)$ of H is the group of automorphisms of H whose restriction to G is the identity.

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